TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 4, April 1999, Pages 1481–1505
S 0002-9947(99)01909-1

INTERSECTIONS OF $\mathbb{Q}$-DIVISORS ON KONTSEVICH'S MODULI SPACE $\overline{M}_{0,n}(\mathbb{P}^r, d)$ AND ENUMERATIVE GEOMETRY

RAHUL PANDHARIPANDE

ABSTRACT. The theory of $\mathbb{Q}$-Cartier divisors on the space of n-pointed, genus 0, stable maps to projective space is considered. Generators and Picard numbers are computed. A recursive algorithm computing all top intersection products of $\mathbb{Q}$-divisors is established. As a corollary, an algorithm computing all characteristic numbers of rational curves in $\mathbb{P}^r$ is proven (including simple tangency conditions). Computations of these characteristic numbers are carried out in many examples. The degree of the 1-cuspidal rational locus in the linear system of degree d plane curves is explicitly evaluated.

0. INTRODUCTION

Kontsevich has defined a new compactification of the space of maps from pointed curves to a fixed projective variety X. Let $\overline{M}_{g,n}(X, \beta)$ denote Kontsevich's compact moduli space of stable maps from n-pointed, genus g curves to X representing the homology class $\beta \in H_2(X, \mathbb{Z})$. The space of stable maps is best behaved when the domain curve is of genus 0 and the target space is a homogeneous variety. In [K-M], [K] the geometry of the space $\overline{M}_{0,n}(X, \beta)$ is used to define Gromov-Witten invariants and to determine an associative quantum cohomology ring of X. In this paper, the geometry of divisors on $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is investigated. As a consequence, a method is developed to calculate classical tangency characteristic numbers of rational curves in projective space.

Let $\mathbb{C}$ be the field of complex numbers. Let $(C, p_1, \ldots, p_n)$ be a connected, reduced, projective, (at worst) nodal curve over $\mathbb{C}$ with n nonsingular marked points $(p_1, \ldots, p_n)$. Let ω_C be the dualizing sheaf of C. An algebraic map

$$\mu : (C, p_1, \ldots, p_n) \to \mathbb{P}^r$$

is *Kontsevich stable* if $\omega_C(p_1 + \cdots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$ is ample on C. Let $\overline{M}_{g,n}(\mathbb{P}^r, d)$ be the coarse moduli space of degree d, Kontsevich stable maps from n-pointed, genus g curves to $\mathbb{P}^r$. In the genus zero case, $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is an irreducible, projective variety with finite quotient singularities. Only the following cases will be considered here:

$$g = 0, \quad r \geq 2, \quad d \geq 0.$$

The stack of Kontsevich stable maps was first defined in [K-M] and [K]. A treatment of the corresponding coarse moduli spaces can also be found in [FP] and [Al].

Received by the editors March 11, 1996.
1991 *Mathematics Subject Classification.* Primary 14N10, 14H10; Secondary 14E99.
Partially supported by an NSF Post-Doctoral Fellowship.

The dimension of $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$ is $m = rd+d+r+n-3$. Let $Pic(\overline{M})$ be the Picard group of line bundles. Let $A_{m-1}(\overline{M})$ be the Chow group of Weil divisors modulo rational equivalence. Since $\overline{M}$ has finite quotient singularities, every Weil divisor is $\mathbb{Q}$-Cartier. Therefore, there is a canonical isomorphism:

$$Pic(\overline{M}) \otimes \mathbb{Q} \stackrel{\sim}{\to} A_{m-1}(\overline{M}) \otimes \mathbb{Q}\ . \tag{1}$$

The first results of the paper concern the dimension and generation of $Pic(\overline{M}) \otimes \mathbb{Q}$. Via the isomorphism (1), Weil divisors determine elements of $Pic(\overline{M}) \otimes \mathbb{Q}$.

Let $I = \{1, 2, \ldots n\}$ be the set of markings (I may be the empty set). The n markings of the moduli problem yield n canonical line bundles $\mathcal{L}_i = \nu_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$ on $\overline{M}$ via the n evaluation maps $\forall i \in I,\ \ \nu_i : \overline{M} \to \mathbb{P}^r$. The *boundary* of $\overline{M}$ is the locus corresponding to maps with reducible domain curves. Since the boundary is of pure codimension 1 in $\overline{M}$, each irreducible component is a Weil divisor. The irreducible components of the boundary are in bijective correspondence with data of weighted partitions $(A \cup B, d_A, d_B)$, where:

(i) $A \cup B$ is a partition of I.
(ii) $d_A + d_B = d$, $d_A \geq 0$, $d_B \geq 0$.
(iii) If $d_A = 0$ (resp. $d_B = 0$), then $|A| \geq 2$ (resp. $|B| \geq 2$).

For example, if $I = \emptyset$, then $A = B = \emptyset$ and the boundary components correspond to positive partitions $d_A + d_B = d$. Let $\triangle$ be the set of components of the boundary.

In case $d \geq 1$, a Weil divisor is obtained on $\overline{M}$ by considering the locus of $\overline{M}$ corresponding to maps meeting a fixed $r-2$ dimensional linear subspace of $\mathbb{P}^r$ (note that $r \geq 2$). Denote the corresponding class in $Pic(\overline{M}) \otimes \mathbb{Q}$ by $\mathcal{H}$. For convenience, let $\mathcal{H} = 0$ in case $d = 0$.

Theorem 1. *$Pic(\overline{M}) \otimes \mathbb{Q}$ is generated by the following elements:*

(i) *If $g = 0$, $r \geq 2$, $d = 0$, then $\{\mathcal{L}_i\} \cup \triangle$ generate $Pic(\overline{M})$.*
(ii) *If $g = 0$, $r \geq 2$, $d \geq 1$, then $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ generate $Pic(\overline{M}) \otimes \mathbb{Q}$.*

If $d = 0$, then (by stability) $n \geq 3$ and $\overline{M}_{0,n}(\mathbb{P}^r, 0) \stackrel{\sim}{=} \overline{M}_{0,n} \times \mathbb{P}^r$, where $\overline{M}_{0,n}$ is the (nonsingular) Mumford-Knudsen space. In this case, $\mathcal{L}_i$ is the pull-back of $\mathcal{O}_{\mathbb{P}^r}(1)$ from the second factor. Therefore, part (i) is a consequence of the boundary generation of $Pic(\overline{M}_{0,n})$.

There is an intersection pairing $A_1(\overline{M}) \otimes Pic(\overline{M}) \to \mathbb{Z}$. Let $Null \subset Pic(\overline{M})$ be the null space with respect to the intersection pairing. Define

$$Num(\overline{M}) = Pic(\overline{M})/Null.$$

By Theorem (1), the classes $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ generate $Num(\overline{M}) \otimes \mathbb{Q}$. The relations between these generators in $Num(\overline{M}) \otimes \mathbb{Q}$ can be determined by calculating intersections with curves. It will be shown that all the relations in $Num(\overline{M}) \otimes \mathbb{Q}$ are obtained from linear equivalences in $Pic(\overline{M}) \otimes \mathbb{Q}$.

Theorem 2. *The canonical map $Pic(\overline{M}) \otimes \mathbb{Q} \to Num(\overline{M}) \otimes \mathbb{Q}$ is an isomorphism. The Picard numbers are $(g = 0, r \geq 2, d \geq 0)$:*

$(n = 0)$, $dim_{\mathbb{Q}}\ Pic(\overline{M}) \otimes \mathbb{Q} = [\frac{d}{2}] + 1$.
$(n \geq 1)$, $dim_{\mathbb{Q}}\ Pic(\overline{M}) \otimes \mathbb{Q} = (d+1) \cdot 2^{n-1} - \binom{n}{2}$.

The main result of this paper concerns the computation of top intersection products in $Pic(\overline{M}) \otimes \mathbb{Q}$.

Theorem 3. *Let $g = 0$, $r \geq 2$, $d \geq 0$. There exists an explicit algorithm for calculating the top dimensional intersection products of the $\mathbb{Q}$-Cartier divisors $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ on $\overline{M}$.*

Consider the space $R(d, r)$ of degree $d \geq 1$ rational curves in $\mathbb{P}^r$ ($r \geq 2$). The dimension of $R(d, r)$ is $rd + r + d - 3$. Classically, the characteristic numbers of $R(d, r)$ are the numbers of degree d rational curves in $\mathbb{P}^r$ passing through α_i general linear spaces of codimension i (for $2 \leq i \leq r$) and tangent to β general hyperplanes, where

$$\sum_{i=2}^{r} (i-1) \cdot \alpha_i + \beta = dim\ R(d, r).$$

The characteristic numbers of rational curves excluding tangencies ($\beta = 0$) have been determined recursively by both symplectic ([R-T]) and algebraic ([K-M]) methods. The divisor in $\overline{M}$ corresponding to the hyperplane tangency condition can be expressed as a linear combination of the classes $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$. The characteristic numbers can then be expressed as top intersection products of $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ on suitably chosen Kontsevich spaces of maps $\overline{M}$. Therefore, all the characteristic numbers can be calculated by Theorem (3).

Theorem 4. *There exists an explicit algorithm for calculating all the characteristic numbers of rational curves in projective space.*

P. Di Francesco and C. Itzykson have modified the methods of [K-M] to determine some ($\beta \neq 0$) characteristic numbers for rational plane curves ([D-I]). The relations they obtain from the WDVV-associativity equations do not suffice to recursively determine all the characteristic numbers for rational plane curves from a finite set of data.

The structure of the paper is as follows. Theorems (1) and (2) are proven in section (1). In section (2), several geometric classes are explicitly computed in $Pic(\overline{M}) \otimes \mathbb{Q}$. These classes are used in the algorithms of Theorems (3) and (4). The algorithms are established in section (3). Section (4) is devoted to calculations of some characteristic numbers of rational curves for small values of (d, r). As a final application, a new formula for cuspidal rational curves is derived in section (4.5). Further computations in $Pic(\overline{M}) \otimes \mathbb{Q}$ involving the canonical class of $\overline{M}$ and adjunction can be found in [P].

The problem of calculating tangency characteristic numbers via Kontsevich's moduli space was suggested to the author by W. Fulton. Conversations with W. Fulton on related topics have been of significant aid. Thanks are due to S. Kleiman and P. Aluffi for mathematical and historical remarks. The remarkable ideas in [K-M] have been a source of inspiration.

1. $Pic(\overline{M}) \otimes \mathbb{Q}$ AND $Num(\overline{M}) \otimes \mathbb{Q}$

1.0. **Summary.** Theorems (1) and (2) are established in sections (1.1) and (1.2) respectively. Since these results are well known for $d = 0$,

$$\overline{M}_{0,n}(\mathbb{P}^r, 0) \cong \overline{M}_{0,n} \times \mathbb{P}^r,$$

the conditions $g = 0$, $r \geq 2$, $d \geq 1$ are assumed throughout sections (1.1) and (1.2).

1.1. **Generators.** The proof of Theorem (1) is divided into four cases depending upon the number n of marked points.

Lemma 1.1.1. *If $n \geq 3$, then $Pic(\overline{M}) \otimes \mathbb{Q}$ is generated by $\triangle \cup \{\mathcal{H}\}$.*

Proof. Let $V = \bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. Let $U \subset \mathbb{P}(V)$ be the Zariski open set corresponding to a well defined (basepoint free) degree d map from $\mathbb{P}^1$ to $\mathbb{P}^r$. The complement of U in $\mathbb{P}(V)$ is of codimension at least $r \geq 2$. There is a universal map

$$\mathbb{P}^1 \times U \to \mathbb{P}^r.$$

Fix the first three marked points to be $0,\ 1,\ \infty \in \mathbb{P}^1$. Let

$$W = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ \setminus \{D_{i,j},\ S_{0,i},\ S_{1,i},\ S_{\infty,i}\},$$

where the product is taken over $n-3$ factors. $D_{i,j}$ is the large diagonal determined by factors i and j. $S_{0,i}$ is the locus where the i^{th} factor is $0 \in \mathbb{P}^1$. $S_{1,i}$, $S_{\infty,i}$ are defined similarly. It follows there is a universal family of Kontsevich stable degree d maps of n-pointed curves:

$$\mathbb{P}^1 \times W \times U \to \mathbb{P}^r.$$

The maps of the family are automorphism-free and distinct. By the universal property, there is an injection $W \times U \to \overline{M}$. A tangent space calculation shows that $W \times U$ is an open set of $\overline{M}$. The complement of $W \times U$ is the boundary of $\overline{M}$. Hence $A_{m-1}(\overline{M})$ is generated by $\triangle$ and $A_{m-1}(W \times U)$. Information about $A_{m-1}(W \times U)$ is obtained from the open inclusion

$$W \times U \subset \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \times \mathbb{P}(V). \tag{2}$$

The Picard group of the right side of (2) is generated by the pull-backs of $\mathcal{O}(1)$ from each factor. The pull-backs from the $\mathbb{P}^1$ factors are trivial on $W \times U$ because of the removal of the loci $S_{0,i}$. Hence, $A_{m-1}(W \times U)$ is generated by $\mathcal{O}_{\mathbb{P}(V)}(1)$. It is easily seen that $\mathcal{H}$ restricted to $W \times U$ is the pull-back of a resultant hypersurface in $\mathbb{P}(V)$. Therefore, $\mathcal{H}$ restricted to $W \times U$ is linearly equivalent to a multiple of $\mathcal{O}_{\mathbb{P}(V)}(1)$. □

There are canonical morphisms $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n-1}(\mathbb{P}^r, d)$ obtained by omitting the last marked point and possibly contracting. Results for $0 \leq n \leq 2$ are obtained via these morphisms.

Lemma 1.1.2. *If $n = 2$, then $Pic(\overline{M}) \otimes \mathbb{Q}$ is generated by $\triangle \cup \{\mathcal{L}_1, \mathcal{L}_2\}$.*

Proof. Let $\overline{N} = \overline{M}_{0,3}(\mathbb{P}^r, d)$ and $\overline{M} = \overline{M}_{0,2}(\mathbb{P}^r, d)$. Fix a hyperplane $H_3 \subset \mathbb{P}^r$. Let $X = \nu_3^{-1}(H_3)$, where ν_3 is the third evaluation map, $\nu_3 : \overline{N} \to \mathbb{P}^r$. There is a map $\rho : X \to \overline{M}$ obtained by omitting the third point (and possibly contracting). The map ρ is surjective and generically finite. Let $Z \subset \overline{M}$ be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is $\mathbb{P}^1$.
(ii.) The images of the marked points $\{1, 2\}$ do not lie in H_3.

It is clear that the complement of Z is the boundary union $\nu_1^{-1}(H_3)$, $\nu_2^{-1}(H_3)$. By the definition of Z, $\rho^{-1}(Z) \to Z$ is a finite, projective morphism. If $A_{m-1}(\rho^{-1}(Z))$ $= 0$, then $A_{m-1}(Z)$ is torsion. To establish the lemma, it therefore suffices to prove that $A_{m-1}(\rho^{-1}(Z)) = 0$.

In the notation of the proof of Lemma (1.1.1), $\rho^{-1}(Z) \subset U \subset \overline{M}_{0,3}(\mathbb{P}^r, d)$. In fact, the following is easily seen:

$$\rho^{-1}(Z) = U \cap L_\infty(H_3) \setminus \{L_0(H_3), L_1(H_3)\}.$$

$L_p(H_3)$ is the hyperplane in U corresponding to maps sending the point $p \in \mathbb{P}^1$ to H_3. $U \cap L_\infty(H_3)$ is an open set of $L_\infty(H_3)$ with complement of codimension at least 2. Hence, $A_{m-1}(U \cap L_\infty(H_3)) = \mathbb{Z}$, generated by the hyperplane class. Since $\rho^{-1}(Z) \subset U \cap L_\infty(H_3)$ is the complement of hyperplanes, the desired conclusion $A_{m-1}(\rho^{-1}(Z)) = 0$ is obtained. □

Lemma 1.1.3. *If $n = 1$, then $Pic(\overline{M}) \otimes \mathbb{Q}$ is generated by $\triangle \cup \{\mathcal{L}_1, \mathcal{H}\}$.*

Proof. Let $\overline{N} = \overline{M}_{0,3}(\mathbb{P}^r, d)$ and $\overline{M} = \overline{M}_{0,1}(\mathbb{P}^r, d)$. Fix two hyperplanes $H_2, H_3 \subset \mathbb{P}^r$. Let $X = \nu_2^{-1}(H_2) \cap \nu_3^{-1}(H_3)$, where ν_2, ν_3 are the second and third evaluation maps on $\overline{N}$. There is a map $\rho : X \to \overline{M}$ obtained by omitting the second and third points. The map ρ is surjective and generically finite. Let $Z \subset \overline{M}$ be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i) The domain curve is $\mathbb{P}^1$.
(ii) The image of the marked point $\{1\}$ does not lie in $H_2 \cup H_3$.
(iii) The map does not pass through the intersection $H_2 \cap H_3$.

The complement of Z is the boundary union $\nu_1^{-1}(H_2)$, $\nu_1^{-1}(H_3)$, and $D_{2,3}$. $D_{2,3}$ is the Weil divisor of maps passing through $H_2 \cap H_3$. $D_{2,3}$ is a divisor rationally equivalent to $\mathcal{H}$. By the definition of Z, $\rho^{-1}(Z) \to Z$ is a finite, projective morphism. As before, it suffices to prove that $A_{m-1}(\rho^{-1}(Z)) = 0$.

Let $S \subset U \subset \overline{M}_{0,3}(\mathbb{P}^r, d)$ be the union of the hyperplane sections $\{L_0(H_2), L_0(H_3)\}$ with the resultant hypersurface of maps meeting $H_2 \cap H_3$. Conditions (i), (ii), and (iii) imply that

$$\rho^{-1}(Z) = U \cap L_1(H_2) \cap L_\infty(H_3) \setminus S.$$

As before, $A_{m-1}(U \cap L_1(H_2) \cap L_\infty(H_3)) = \mathbb{Z}$, generated by the hyperplane class. S is a union of hyperplane classes and multiples of hyperplane classes. Hence, $A_{m-1}(\rho^{-1}(Z)) = 0$. □

Lemma 1.1.4. *If $n = 0$, then $Pic(\overline{M}) \otimes \mathbb{Q}$ is generated by $\triangle \cup \{\mathcal{H}\}$.*

Proof. Let $\overline{N} = \overline{M}_{0,3}(\mathbb{P}^r, d)$ and $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^r, d)$. Fix three general hyperplanes $H_1, H_2, H_3 \subset \mathbb{P}^r$. Let $X = \nu_1^{-1}(H_1) \cap \nu_2^{-1}(H_2) \cap \nu_3^{-1}(H_3)$, where the ν_i are evaluation maps on $\overline{N}$. There is a map $\rho : X \to \overline{M}$ obtained by omitting the marked points. The map ρ is surjective and generically finite. Let $Z \subset \overline{M}$ be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i) The domain curve is $\mathbb{P}^1$.
(ii) The map does not pass through the intersections $H_1 \cap H_2$, $H_1 \cap H_3$, or $H_2 \cap H_3$.

The complement of Z is the boundary union $D_{1,2}$, $D_{1,3}$, and $D_{2,3}$. By the definition of Z, $\rho^{-1}(Z) \to Z$ is a finite, projective morphism. As before, it suffices to prove that $A_{m-1}(\rho^{-1}(Z)) = 0$.

Let $S \subset U \subset \overline{M}_{0,3}(\mathbb{P}^r, d)$ be the union of the three resultant hypersurfaces of maps meeting $H_1 \cap H_2$, $H_1 \cap H_3$, and $H_2 \cap H_3$. Let $I \subset U$ be the hyperplane intersection defined by $I = U \cap L_0(H_1) \cap L_1(H_2) \cap L_\infty(H_3)$. Conditions (i) and (ii) imply that

$$\rho^{-1}(Z) = I \setminus S \cap I.$$

Note that $S \cap I$ contains the intersections of the following hyperlanes with I:

$$\{L_0(H_2), L_0(H_3), L_1(H_1), L_1(H_3), L_\infty(H_1), L_\infty(H_2)\}.$$

As before, $A_{m-1}(U \cap I) = \mathbb{Z}$, generated by the hyperplane class. Since $S \cap I$ is a union of hyperplane classes and multiples of hyperplane classes, $A_{m-1}(\rho^{-1}(Z)) = 0$. □

Lemmas (1.1.1) - (1.1.4) yield Theorem (1).

1.2. **Relations.** Curves in $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$ are easily found. The following construction will be required for the calculations below. Let C be a nonsingular, projective curve. Let $\pi : S = \mathbb{P}^1 \times C \to C$. Select n sections $s_1, \ldots, s_n$ of π. A point $x \in S$ is an *intersection point* if two or more sections contain x. Let $\overline{N}$ be a line bundle on S of type (d,k), where k is very large. Let $z_l \in H^0(S, \mathcal{N})$ $(0 \leq l \leq r)$ determine a rational map $\mu : S \dashrightarrow \mathbb{P}^r$ with simple base points. A point $y \in S$ is a *simple base point* of degree $1 \leq e \leq d$ if the blow-up of S at y resolves μ locally at y and the resulting map is of degree e on the exceptional divisor E_y. The set of *special points* of S is the union of the intersection points and the simple base points. Three conditions are required:

(1) There is at most one special point in each fiber of π.
(2) The sections through each intersection point x have distinct tangent directions at x.
(3) (i) $d = 0$. No $n-1$ sections pass through a point $x \in S$.
 (ii) $d > 0$. If at least $n-1$ sections pass through a point $x \in S$, then x is not a simple base point of degree d.

Condition (3.ii) implies there are no simple base points of degree d if $n = 0$ or 1. Let $\overline{S}$ be the blow-up of S at the special points. It is easily seen that $\overline{\mu} : \overline{S} \to \mathbb{P}^r$ is a Kontsevich stable family of n-pointed, genus 0 curves over C. Condition (2) ensures that the strict transforms of the sections are disjoint. Condition (3) implies Kontsevich stability. There is a canonical morphism $C \to \overline{M}$. Condition (1) implies C intersects the boundary components transversally.

Lemma 1.2.1. *The following independence results hold for the elements $\{\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2\}$:*

(i) *The element $\mathcal{H}$ is not contained in the linear span of $\triangle$ in $Pic(\overline{M}) \otimes \mathbb{Q}$.*
(ii) *If $n = 1$, then $\{\mathcal{H}, \mathcal{L}_1\}$ are independent modulo $\triangle$.*
(iii) *If $n = 2$, then $\{\mathcal{L}_1, \mathcal{L}_2\}$ are independent modulo $\triangle$.*

Proof. Consider $\pi : S = \mathbb{P}^1 \times C \to C$ with n trivial sections. There are no intersection points. Let $\mathcal{N}$, $z_l \in H^0(S, \mathcal{N})$ be such that μ has no base points (note: since $r \geq 2$, this is easily accomplished). $\mathcal{N}$ has degree type (d,k). For each component $K \in \triangle$, $C \cdot K = 0$. A simple calculation yields $C \cdot \mathcal{H} = \mathcal{N} \cdot \mathcal{N} = 2dk$. Hence $\mathcal{H}$ is not contained in the span of $\triangle$.

Consider $\pi : S = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let s be the trivial section; let s' be the diagonal section. Let $\mu : S \to \mathbb{P}^r$ be a base point free map of type (d,k). The two sections s, s' determine two maps $\tau, \tau' : \mathbb{P}^1 \to \overline{M}_{0,1}(\mathbb{P}^r, d)$. Intersection via τ yields

$$\mathbb{P}^1 \cdot \mathcal{H} = 2dk, \quad \mathbb{P}^1 \cdot \mathcal{L}_1 = k.$$

Intersection via τ' yields

$$\mathbb{P}^1 \cdot \mathcal{H} = 2dk, \quad \mathbb{P}^1 \cdot \mathcal{L}_1 = d + k.$$

In both cases $\mathbb{P}^1 \cdot K = 0$ for any $K \in \triangle$. Therefore $\{\mathcal{H}, \mathcal{L}_1\}$ are independent modulo $\triangle$ in $Pic(\overline{M}) \otimes \mathbb{Q}$ for $n = 1$.

In the $n = 2$ case, twisted families must be considered. Let $E(a, b)$ be the rank two bundle $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ over $\mathbb{P}^1$. Let $S(a, b) = \mathbb{P}(E(a, b))$. Let

$$\mathcal{N} = \mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k)).$$

For large k, let $\mu : S(a, b) \to \mathbb{P}^r$ be a base point free map. The sub-bundles $\mathcal{O}(a)$, $\mathcal{O}(b)$ define sections s_1 and s_2. There is an induced map $\mathbb{P}^1 \to \overline{M}_{0,2}(\mathbb{P}^r, d)$. A calculation yields

$$\mathbb{P}^1 \cdot \mathcal{L}_1 = -ad + k, \quad \mathbb{P}^1 \cdot \mathcal{L}_2 = -bd + k.$$

As before, $\mathbb{P}^1 \cdot K = 0$ for any $K \in \triangle$. It follows that $\{\mathcal{L}_1, \mathcal{L}_2\}$ are independent modulo $\triangle$ in $Pic(\overline{M}) \otimes \mathbb{Q}$ for $n = 2$. □

If $n \geq 1$, let $\triangle_i \subset \triangle$ be the subset of boundary components $(A \cup B, d_A, d_B)$ with marking partition $|A| + |B| = n$ equal to the partition $i + (n - i) = n$. There is a disjoint union

$$\triangle = \bigcup_{i=0}^{[\frac{n}{2}]} \triangle_i.$$

Let $\triangle' = \triangle \setminus (\triangle_0 \cup \triangle_1)$.

Lemma 1.2.2. *The independence of $\triangle_0$, $\triangle_1$ is established by the following results.*

(i) *If $n = 0$, then $\triangle_0 = \triangle$ is a set of linearly independent elements of $Pic(\overline{M}) \otimes \mathbb{Q}$.*

(ii) *If $n = 1$, then $\triangle_0 = \triangle_1 = \triangle$ is a set of linearly independent elements of $Pic(\overline{M}) \otimes \mathbb{Q}$.*

(iii) *If $n \geq 2$, then $\triangle_0 \cup \triangle_1$ is a set of linearly independent elements of $Pic(\overline{M}) \otimes \mathbb{Q}$. Moreover, the span of $\triangle_0 \cup \triangle_1$ does not intersect the span of $\triangle'$ in $Pic(\overline{M}) \otimes \mathbb{Q}$.*

Proof. Let $\pi : S = \mathbb{P}^1 \times C \to C$ be as above with n trivial sections. Let $\mathcal{N}$ be a line bundle on S of degree type (d, k). For large degrees k, the simple base points of μ of degree $1 \leq e \leq d$ can be selected arbitrarily satisfying conditions (1) and (3). For suitable choices of simple base points and base point degrees on S, the classes in assertions (i-iii) can be seen to be independent in $Num(\overline{M}) \otimes \mathbb{Q}$. Therefore, the classes are independent in $Pic(\overline{M}) \otimes \mathbb{Q}$. □

In case $n = 0$, $\triangle \cup \{\mathcal{H}\}$ is a basis of $Pic(\overline{M}) \otimes Q$ via Lemmas (1.1.4), (1.2.1), and (1.2.2). For $1 \leq n \leq 3$, $\triangle_0 \cup \triangle_1 = \triangle$. Hence, the lemmas show that the generators of section (1.1) are also bases for $1 \leq n \leq 3$. The Picard numbers of Theorem (2) can be verified for $0 \leq n \leq 3$.

For $n \geq 4$, let $\overline{M}_{0,n}$ be the Mumford-Knudsen moduli space of n-pointed, genus 0 curves. The boundary components of $\overline{M}_{0,n}$ correspond bijectively to partitions $A \cup B$ of $P = \{1, 2, \ldots, n\}$ such that $|A|, |B| \geq 2$. The boundary components generate $Pic(\overline{M}_{0,n})$. The three boundary components of $\overline{M}_{0,4}$ are linearly equivalent. A four element subset $Q \subset P$ induce a natural map $\overline{M}_{0,n} \to \overline{M}_{0,Q}$. The pull-backs of the basic boundary linear equivalences on $\overline{M}_{0,Q}$ induce boundary linear equivalences on $\overline{M}_{0,n}$. The relations among the boundary components of $\overline{M}_{0,n}$ are generated by these pull-back linear equivalences as Q varies among all four element subsets of P. $Pic(\overline{M}_{0,n})$ is a free group of rank

$$2^{n-1} - \binom{n-1}{2} - n.$$

Since there are

$$\frac{2^n - 2 - 2n}{2} = 2^{n-1} - 1 - n$$

boundary components of $\overline{M}_{0,n}$, it follows there are $\binom{n-1}{2} - 1$ independent relations among the boundary components. Finally, $Pic(\overline{M}_{0,n}) \stackrel{\sim}{=} Num(\overline{M}_{0,n})$. See [Ke] for proofs of these results.

Let $n \geq 4$. There is a canonical morphism $\eta : \overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$. The η pull-back of a boundary component of $\overline{M}_{0,n}$ is a non-empty, multiplicity-free sum of boundary components $\triangle'$ of M:

$$\eta^{-1}\big((A \cup B)\big) = \sum_{d_A + d_B = d} (A \cup B, d_A, d_B).$$

Lemma 1.2.3. *The relations among the boundary components $\triangle'$ in $Pic(\overline{M}) \otimes \mathbb{Q}$ are the η pull-backs of the relations among the boundary components of $\overline{M}_{0,n}$.*

Proof. Let $\pi : S = \mathbb{P}^1 \times C \to C$ be a family with n sections. Let $\mu : S \dashrightarrow \mathbb{P}^r$ be a rational map with simple base points obtained from a line bundle of degree type (d, k). Suppose the special points satisfy (1), (2), and

- (3′) An intersection point lies on at most $n - 2$ sections.
- (4) Every simple base point is an intersection point.

Note that condition (3′) implies condition (3). For large k, the simple base points may be selected arbitrarily (with arbitrary degree) among the intersection points. Let $\overline{S}$ be the blow-up of S at the special points; let $\lambda : C \to \overline{M}$ be the induced curve. By condition (3′), the family $\overline{S} \to C$ with the strict transforms of the sections is a flat family of stable, n-pointed, genus 0 curves. The induced morphism $\gamma : C \to \overline{M}_{0,n}$ is simply $\gamma = \eta \circ \lambda$.

Suppose $\sum_{K \in \triangle'} c_K K = 0$ is a relation in $Pic(\overline{M}) \otimes \mathbb{Q}$ ($c_B \in \mathbb{Q}$). Let

$$K = (A \cup B, d_A, d_B) \in \triangle'.$$

Let $(A \cup B)$ be the corresponding boundary component of $\overline{M}_{0,n}$. The set theoretic intersection $C \cdot K$ is the subset of $C \cdot (A \cup B)$ with simple base points of the correct degree. Since the simple base points can be assigned arbitrary degrees, the coefficient c_K must depend only on the partition $(A \cup B)$ and not on the weights d_A, d_B. It now follows that the relation $\sum_{K \in \triangle'} c_K \cdot K = 0$ must be the η pull-back of a boundary relation in $\overline{M}_{0,n}$. □

In particular, it follows that there are $\binom{n-1}{2} - 1$ independent relations among the boundary components $\triangle'$. For $n \geq 4$,

$$|\,\triangle\,| = d + dn + |\,\triangle'\,|,$$

$$|\,\triangle'\,| = (d+1) \cdot (2^{n-1} - 1 - n).$$

By Lemmas (1.1.1), (1.2.1), (1.2.2), (1.2.3), the Picard number of $\overline{M}$ ($n \geq 4$) is

$$dim\ Pic(\overline{M}) \otimes \mathbb{Q} = (d+1) \cdot 2^{n-1} - \binom{n}{2}.$$

All the numerical relations are obtained from linear equivalences. The proof of Theorem (2) is complete.

2. Computations in $Pic(\overline{M}) \otimes \mathbb{Q}$

2.1. **The Universal Curve and $\pi_*(c_1(\omega_\pi)^2)$.** Classes of certain canonical elements in $Pic(\overline{M}) \otimes \mathbb{Q}$ will be computed via intersections with curves. These computations will be used in the proof of Theorem (3). In order to use the coarse moduli space throughout, an automorphism result is required.

Lemma 2.1.1. *Let $g = 0$, $r \geq 2$, $d \geq 0$. The locus of Kontsevich stable maps in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ with nontrivial automorphisms is of codimension at least 2 except in one case: $\overline{M}_{0,0}(\mathbb{P}^2, 2)$.*

Proof. The assertion follows from naive dimension estimates. If $d = 0$ or 1, there are no stable maps with nontrivial automorphisms. Let $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$ ($d \geq 2, r \geq 2$). Recall that $dim\ \overline{M} = rd + d + r + n - 3$. Certainly the generic elements of the boundary components are automorphism-free. Let $A \subset \overline{M}$ be the locus of non-boundary, stable maps with nontrivial automorphisms. If a map $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ with n distinct marked points has a nontrivial automorphism, μ must be a $k \geq 2$ to 1 map, for some $k \geq 2$. For fixed $2 \leq k \leq d$, the map μ moves in a family of dimension at most

$$(r+1) \cdot (\frac{d}{k} + 1) - 1 - 3 + 2 \cdot (k+1) - 1 - 3 = (rd + d) \cdot \frac{1}{k} + r - 3 + 2k - 2.$$

The n marked points must be fixed points of the nontrivial automorphism and hence move in a zero dimensional family for each μ. A calculation yields

$$\begin{aligned} dim\ \overline{M} - dim\ A &\geq (rd + d) \cdot (1 - \frac{1}{k}) + n - 2k + 2 \\ &= rd + d + n + 2 - \frac{rd + d + 2k^2}{k}. \end{aligned}$$

A study of the function $(rd + d + 2k^2)/k$ for $2 \leq k \leq d$ shows that the maximum value must be attained at the end points $k = 2, d$. If $k = 2$, then

$$rd + d - \frac{rd + d + 8}{2} = (r+1)\frac{d}{2} - 4 \geq 0$$

except when $r = 2$, $d = 2$. If $k = d$, then

$$rd + d - \frac{rd + d + 2d^2}{d} = (r-1)(d-1) - 2 \geq 0$$

except when $r = 2$, $d = 2$. Hence, A is of codimension at least 2 all cases except $\overline{M}_{0,0}(\mathbb{P}^2, 2)$. ☐

Since $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics (see [FP]), its intersection theory is well known. In the sequel, it will be assumed that $(g, n, r, d) \neq (0, 0, 2, 2)$. Let $\overline{M}^* \subset \overline{M}$ denote the automorphism-free locus. There is a universal Kontsevich stable family of maps over $\overline{M}^*$:

$$\pi : U^* \to \overline{M}^*$$

with sections $s_1, s_2 \ldots, s_n$ and a morphism

$$\mu : U^* \to \mathbb{P}^r.$$

See [FP] for details. Let ω_π be the relative dualizing sheaf of π. Since the complement of $\overline{M}^*$ is of codimension at least 2 in $\overline{M}$, the following are well-defined elements of $A_{m-1}(\overline{M})$ and therefore of $Pic(\overline{M}) \otimes \mathbb{Q}$:

$$\pi_*(c_1(\omega_\pi)^2), \quad \pi_*(s_i^2). \tag{3}$$

Since $Pic(\overline{M}) \otimes \mathbb{Q} \cong Num(\overline{M}) \otimes \mathbb{Q}$, explicit expressions of the classes (3) in terms of the generators $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ can be found by calculating intersection products with curves in $\overline{M}$. The methods of section (1.2) will be used to determine curves in $\overline{M}$. First consider $\pi_*(c_1(\omega_\pi)^2)$:

Lemma 2.1.2. *For $g = 0$, $r \geq 2$, $d \geq 0$,*

$$\pi_*(c_1(\omega_\pi)^2) = -\sum_{K \in \triangle} K \quad \text{in } Pic(\overline{M}) \otimes \mathbb{Q}.$$

Proof. Let $\pi : S \to C$ be a projective bundle of rank 1 over a nonsingular curve C. Let ω_π be the relative dualizing sheaf. A simple computation yields

$$\pi_*(c_1(\omega_\pi)^2) = 0$$

in $Num(C)$. Let $\rho : \overline{S} \to S$ be the blow-up at k points in distinct fibers of π. Let $\overline{\pi} : \overline{S} \to C$ be the composition. Then

$$\omega_{\overline{\pi}} = \rho^*(\omega_\pi) + \sum_{i=1}^{k} E_i,$$

where the E_i are the exceptional divisors of ρ. Hence,

$$\pi_*(c_1(\omega_{\overline{\pi}})^2) = -k$$

in $Num(C)$. By considering curves $C \to \overline{M}^*$ and the pull-back of U^*, it follows that $\pi_*(c_1(\omega_\pi)^2) = -\sum_{K \in \triangle} K$ in $Num(\overline{M}) \otimes \mathbb{Q}$. By Theorem (2), the lemma is proven. □

2.2. **The Class $\pi_*(s_1^2)$.** The determination of the class $\pi_*(s_1^2)$ is surprisingly different in the cases $d = 0$ and $d \geq 1$. If $d = 0$, it suffices to determine $\pi_*(s_1^2)$ for the universal family over $\overline{M}_{0,n}$. Let $\triangle$ be the set of boundary components of $\overline{M}_{0,n}$. There is a partition of $\triangle$ with respect to the first marking. For $2 \leq j \leq n-2$, let $\triangle_j^1 \subset \triangle$ be defined by:

$$(A \cup B) \in \triangle_j^1 \quad \text{if and only if} \quad 1 \in A,\ |A| = j.$$

There is a disjoint union

$$\triangle = \bigcup_{j=2}^{n-2} \triangle_j^1.$$

Let $K_j^1 = \sum_{K \in \triangle_j^1} K$.

Lemma 2.2.1. *The class $\pi_*(s_1^2)$ is expressed in $Pic(\overline{M}_{0,n}) \otimes \mathbb{Q}$ by*

$$\pi_*(s_1^2) = -\frac{1}{\binom{n-1}{2}} \cdot \sum_{j=2}^{n-2} \binom{n-j}{2} K_j^1. \tag{4}$$

Proof. The proof is by intersections with curves in $\overline{M}_{0,n}$. Let $S = \mathbb{P}^1 \times C$ be a family with n sections $s_1, \ldots, s_n$. Let s_1 be of degree type $(1, q)$. For $2 \leq i \leq n$, let s_i be of type $(1, p_i)$. As usual, assume the blow-up $\overline{S}$ up of S at the intersection points yields a family of stable, n-pointed curves over C with at most one exceptional divisor in each fiber. Let $\lambda : C \to \overline{M}_{0,n}$ be the induced map. It will be checked that the left and right sides of (4) have the same intersection with C.

A point of $C \cdot K^1_j$ can arise in exactly two cases. First, an intersection point of j sections including s_1 can be blown-up. Second, an intersection point of $n-j$ sections not including s_1 can be blown-up. Let

$$C \cdot K^1_j = x_j + y_j.$$

where x_j, y_j are the number of instances of the first and second cases respectively. Let $\overline{s}_1$ be the strict transform of s_1 in $\overline{S}$. The intersection of C with the left side of (4) is

$$\overline{\pi}_*(\overline{s}_1^2) = 2q - \sum_{j=2}^{n-2} x_j.$$

For $2 \leq i \leq n$, s_i intersects s_1 in $q + p_i$ points. The following equation is easily obtained by analyzing intersection points contained in s_1:

$$(n-1)q + \sum_{i=2}^{n} p_i = \sum_{j=2}^{n-2} (j-1) x_j. \tag{5}$$

Similarly, the number of intersections of the sections $2 \leq i \leq n$ among themselves is $(n-2) \cdot \sum_{i=2}^n p_i$. Analysis of intersection points not contained in s_1 yields

$$(n-2) \cdot \sum_{i=2}^{n} p_i = \sum_{j=2}^{n-2} \binom{j-1}{2} x_j + \binom{n-j}{2} y_j. \tag{6}$$

Via equations (5) and (6),

$$\begin{aligned} &\binom{n-1}{2} \cdot (2q - \sum_{j=2}^{n-2} x_j) \\ &= \sum_{j=2}^{n-2} \left((n-2)(j-1) - \binom{j-1}{2} - \binom{n-1}{2} \right) x_j - \binom{n-j}{2} y_j \\ &= - \sum_{j=2}^{n-2} \binom{n-j}{2} (x_j + y_j). \end{aligned}$$

Hence both sides of equation (4) have the same intersection numbers with C. Let D be any nonsingular curve in $\overline{M}_{0,n}$ which intersects the boundary transversely. The universal family over D can be blown-down to a projective bundle $\pi : T \to D$. The above calculation covers the case where $T = \mathbb{P}^1 \times D$. The general case (in which T is *any* $\mathbb{P}^1$-bundle) is identical. Since $A^1(\overline{M}_{0,n})$ is spanned by curves meeting the boundary transversely and

$$Pic(\overline{M}_{0,n}) \otimes \mathbb{Q} \stackrel{\sim}{=} Num(\overline{M}_{0,n}) \otimes \mathbb{Q},$$

Lemma (2.2.1) is proved. □

Consider now the case $d \geq 1$. Let $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$, where $d \geq 1$, $n \geq 1$. Let 1 be the first marking. There is another partition of $\triangle$ with respect to the first marking depending upon the degree. For $0 \leq j \leq d$, let $\triangle^{1,j} \subset \triangle$ be defined by

$$(A \cup B, d_A, d_B) \in \triangle^{1,j} \quad \text{if and only if} \quad 1 \in A, \ d_A = j.$$

Note that if $n = 1$, then $\triangle^{1,0}, \triangle^{1,d} = \emptyset$. If $n = 2$, $\triangle^{1,d} = \emptyset$. In all other cases $\triangle^{1,j} \neq \emptyset$. There is a disjoint union

$$\triangle = \bigcup_{j=0}^{d} \triangle^{1,j}.$$

Let $K^{1,j} = \sum_{K \in \triangle^{1,j}} K$. Let $K^{1,j} = 0$ if $\triangle^{1,j} = \emptyset$.

Lemma 2.2.2. *In case $d \geq 1$, the class $\pi_*(s_1^2)$ is expressed in $Pic(\overline{M}) \otimes \mathbb{Q}$ by*

$$\pi_*(s_1^2) = -\frac{1}{d^2}\mathcal{H} + \frac{2}{d}\mathcal{L}_1 - \sum_{j=0}^{d} \frac{(d-j)^2}{d^2} K^{1,j}. \tag{7}$$

Proof. The proof is by intersections with curves in $\overline{M}$. Let $\pi : S = \mathbb{P}^1 \times C \to C$ be a family with n sections $s_1, \ldots, s_n$. Let s_1 be of degree type $(1, q)$. Let $\mu : S \dashrightarrow \mathbb{P}^r$ be a rational map with simple base points obtained from a line bundle of degree type (d, k). Let conditions (1), (2), (3′), (4) of section (1.2) be satisfied. Let $\overline{S} \to S$ be the blow-up at the special points. Let $\lambda : C \to \overline{M}$ be the induced map. It will be checked that the left and right sides of (7) have the same intersection with C. As in the proof of Lemma (2.2.1), the case in which $S \to C$ is a nontrivial $\mathbb{P}^1$-bundle must be checked. The algebra for this more general case is similar and yields the same result.

A point of $C \cdot K^{1,j}$ can arise in exactly two cases. First, a simple base point of degree j contained in s_1 can be blown-up. Second, a simple base point of degree $d - j$ not contained in s_1 can be blown-up. Let

$$C \cdot K^{1,j} = x_j + y_j,$$

where x_j, y_j are the number of instances of the first and second cases respectively. Let $\overline{s}_1$ be the strict transform of the section s_1 to $\overline{S}$. The intersection of C with the left side of (7) is given by

$$\overline{\pi}_*(\overline{s}_1^2) = 2q - \sum_{j=0}^{d} x_j.$$

A straightforward computation yields

$$C \cdot \mathcal{H} = 2dk - \sum_{j=0}^{d} j^2 x_j - \sum_{j=0}^{d} (d-j)^2 y_j,$$

$$C \cdot \mathcal{L}_1 = dq + k - \sum_{j=0}^{d} j x_j.$$

The equality of the intersection of C with the left and right sides of (7) is now a matter of simple algebra. □

2.3. **The Class $\mathcal{T}$.** Let $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$, $d \geq 2$. Let $H \subset \mathbb{P}^r$ be a hyperplane. A tangency Weil divisor $\mathcal{T}_H \subset \overline{M}$ is defined as follows. Let $W_H \subset \overline{M}_{0,0}(\mathbb{P}^r, d)$ be the open locus of maps $\mu : C \to \mathbb{P}^r$, where $\mu^{-1}(H)$ is a subscheme of d reduced points of $C_{nonsing}$. Let ρ: $\overline{M} \to \overline{M}_{0,0}(\mathbb{P}^r, d)$ be the contraction map. Let $\mathcal{T}_H$ be the complement of $\rho^{-1}(W_H)$.

It must be shown that $\mathcal{T}_H$ is of pure codimension 1 in $\overline{M}$. Let $M_H \subset \overline{M}$ be the open locus of maps $\mu : C \to \mathbb{P}^r$ satisfying

$$\forall x \in \mu^{-1}(H), \quad x \in C_{nonsing} \text{ and } d\mu_x \neq 0.$$

The intersection $\mathcal{T}_H \cap M_H$ corresponds to geometric tangencies and is certainly of pure codimension 1 in M_H ($d \geq 2$). The complement $\overline{M} \setminus M_H$ is of codimension 2 in $\overline{M}$. It is not hard to see that the closure of $\mathcal{T}_H \cap M_H$ in $\overline{M}$ contains the complement $\overline{M} \setminus M_H$. Therefore, $\mathcal{T}_H$ is a Weil divisor.

For $0 \leq j \leq [\frac{d}{2}]$, define $\triangle^j \subset \triangle$ as follows. A boundary component $(A \cup B, d_A, d_B) \in \triangle^j$ if and only if the degree partition $d_A + d_B = d$ equals the partition $j + (d-j) = d$. Let $K^j = \sum_{K \in \triangle^j} K$.

Lemma 2.3.1. *The class of $\mathcal{T}$ can be expressed in $Pic(\overline{M}) \otimes \mathbb{Q}$ by*

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{j=0}^{[\frac{d}{2}]} \frac{j(d-j)}{d} K^j. \tag{8}$$

Proof. Let S, μ, $\overline{S}$, $\lambda : C \to \overline{M}$ be exactly as in the proof of Lemma (2.2.2). It will be checked that the left and right sides of (8) have the same intersection with C. As in the proof of Lemma (2.2.1), the case in which $S \to C$ is a nontrivial $\mathbb{P}^1$-bundle must be checked. The algebra for this more general case is similar and yields the same result.

As before, a point of the intersection $C \cdot K^j$ can arise in two cases. A simple point of degree j or $d-j$ can be blown-up. Let

$$C \cdot K^j = x_j + y_j,$$

where x_j and y_j are the number instances of the first and second case respectively. Let E_{x_j} be the union of the x_j exceptional divisors in $\overline{S}$ obtained from the x_j points of $C \cdot K^j$. Let E_{y_j} be defined similarly.

First, the intersection $C \cdot \mathcal{T}$ is calculated. A general element of $\overline{\mu}^*(\mathcal{O}_{\mathbb{P}}(1))$ is a nonsingular curve D in the linear series $(d,k) - \sum_j jE_{x_j} - \sum_j (d-j)E_{y_j}$. Adjunction yields

$$2g_D - 2 = d(2g_C - 2) + 2dk - 2k - \sum_{j=0}^{[\frac{d}{2}]} j(j-1)x_j - \sum_{j=0}^{[\frac{d}{2}]} (d-j)(d-j-1)y_j.$$

Since D is a d sheeted cover of C, the Riemann-Hurwitz formula determines the ramifications:

$$C \cdot \mathcal{T} = 2dk - 2k - \sum_{j=0}^{[\frac{d}{2}]} j(j-1)x_j + (d-j)(d-j-1)y_j.$$

$C \cdot \mathcal{H}$ is simply D^2. Hence

$$C \cdot \mathcal{H} = 2dk - \sum_{j=0}^{[\frac{d}{2}]} j^2 x_j + (d-j)^2 y_j.$$

Again, an algebraic computation yields the equality of the intersections of the left and right sides with C. $\square$

3. Intersections of $\mathbb{Q}$-divisors

3.1. **Intersections of the Classes $\{\mathcal{L}_i\} \cup \{\mathcal{H}\}$.** The top dimensional intersection products on $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$ of the classes $\{\mathcal{L}_i\}$ are algorithmically determined by the First Reconstruction Theorem [K-M]. These top classes are computed recursively in d and n. The algorithm requires one initial value: the number of lines in $\mathbb{P}^r$ through two points. The top intersection products of $\{\mathcal{L}_i\}$ are exactly the characteristic numbers ($\beta = 0$) of rational curves in $\mathbb{P}^r$.

Top dimensional intersections of the classes $\{\mathcal{L}_i\} \cup \{\mathcal{H}\}$ are also characteristic numbers of rational curves in $\mathbb{P}^r$. Each factor of $\mathcal{H}$ is a codimension-1 characteristic condition. For example, if $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^2, 3)$, then $\mathcal{H}^8$ equals the number of rational plane cubics through 8 general points. If $\overline{M} = \overline{M}_{0,2}(\mathbb{P}^3, 4)$, then $c(\mathcal{L}_1)^3 \cdot c(\mathcal{L}_2)^3 \cdot \mathcal{H}^{12}$ equals the number of rational space quartics passing through 2 general points and meeting 12 general lines. In sections (3.2)-(3.3), a method of computing all top intersection products in $Pic(\overline{M}) \otimes \mathbb{Q}$ is determined. In section (3.4), the relation to enumerative geometry is proven.

3.2. **Boundary Components.** Let $K = (A \cup B, d_A, d_B)$ be a boundary component of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. Let $\overline{M}_A = \overline{M}_{0,|A|+1}(\mathbb{P}^r, d_A)$ and $\overline{M}_B = \overline{M}_{0,|B|+1}(\mathbb{P}^r, d_B)$. Let the additional markings be p_A and p_B respectively. Let $e_A : \overline{M}_A \to \mathbb{P}^r$ and $e_B : \overline{M}_B \to \mathbb{P}^r$ be the evaluation maps obtained from the markings p_A and p_B. Let τ_A, τ_B be the projections from $\overline{M}_A \times \overline{M}_B$ to the first and second factors respectively. Let $\tilde{K} = \overline{M}_A \times_{\mathbb{P}^r} \overline{M}_B$ be the fiber product with respect to the evaluation maps e_A, e_B. $\tilde{K} \subset \overline{M}_A \times_{\mathbb{C}} \overline{M}_B$ is the closed subvariety $(e_A \times_{\mathbb{C}} e_B)^{-1}(D)$, where $D \subset \mathbb{P}^r \times \mathbb{P}^r$ is the diagonal. $\tilde{K}$ is easily seen to be an irreducible, normal, projective variety with finite quotient singularities. These results follow, for example, from the local construction given in [FP]. The class of $\tilde{K}$ in $\overline{M}_A \times \overline{M}_B$ can be computed by the pull-back of the Künneth decomposition of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$:

$$[\tilde{K}] = \sum_{i=0}^{r} \tau_A^*(c_1(\mathcal{L}_A)^i) \cdot \tau_B^*(c_1(\mathcal{L}_B)^{r-i}), \tag{9}$$

where $\mathcal{L}_A$, $\mathcal{L}_B$ are the line bundles on $\overline{M}_A$, $\overline{M}_B$ induced by the marking p_A, p_B.

There is a natural map $\psi : \tilde{K} \to K$. The set theoretic description of ψ is clear: $\psi([\mu_A], [\mu_B])$ is the moduli point of the map obtained by gluing maps μ_A, μ_B along the markings p_A, p_B. It is not hard to define ψ algebraically. ψ is a birational morphism except when $n = 0$ and $d_A = d_B = d/2$. In the latter case, ψ is generically 2-1.

The pull-backs of the classes $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ on $\overline{M}$ to $\tilde{K}$ are determined in the following manner. Let $\mathcal{H}_A$, $\mathcal{H}_B$ be the codimension-2 plane incidence classes on $\overline{M}_A$, $\overline{M}_B$. Clearly,

$$\psi^*(\mathcal{H}) = (\tau_A^*(\mathcal{H}_A) + \tau_B^*(\mathcal{H}_B)) \mid_{\tilde{K}}. \tag{10}$$

Let P be the marking set of $\overline{M}$. Each $i \in P$ is either in A or B. It follows that

$$\psi^*(\mathcal{L}_i) = \tau_A^*(\mathcal{L}_i) \mid_{\tilde{K}}, \quad \psi^*(\mathcal{L}_i) = \tau_B^*(\mathcal{L}_i) \mid_{\tilde{K}} \tag{11}$$

in case $i \in A$, $i \in B$ respectively.

Let $T = (A' \cup B', d_{A'}, d_{B'})$ be a boundary component of $\overline{M}$ *not* equal to K. T intersects K exactly when one of the following two conditions holds:

(i) There exist a subset $C \subset A$ and an integer d_C such that

$$((A \setminus C) \cup (B \cup C), d_A - d_C, d_B + d_C) = T.$$

(ii) There exist a subset $C \subset B$ and an integer d_C such that

$$((A \cup C) \cup (B \setminus C), d_A + d_C, d_B - d_C) = T.$$

The pull-back $\psi^*(T)$ has the following expression:

$$\psi^*(T) = \sum_{C,d_c} \tau_A^*(A \cup (C \cup \{p_A\}), d_A, d_C) \mid_{\tilde{K}} + \sum_{C,d_c} \tau_B^*(B \cup (C \cup \{p_B\}), d_B, d_C) \mid_{\tilde{K}}. \tag{12}$$

The sums on the right are taken over subsets C and degrees d_c that satisfy (i) and (ii) above, respectively. The main point is that distinct boundary divisors have transverse (if nonempty) intersections in the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ ([K]). This can be seen as a property inherited from the Mumford-Knudsen space $\overline{M}_{0,m}$ by the local construction given in [FP]. Since the automorphism loci of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and the boundary component $(A \cup B, d_A, d_B)$ are of codimension at least two in $\overline{M}_{0,n}$ and in $(A \cup B, d_A, d_B)$ respectively, the transverse intersection property descends to the coarse moduli space.

Let $\omega_{\pi A}$, $\omega_{\pi B}$ denote the relative dualizing sheaves of the universal families over $\overline{M}^*_A$, $\overline{M}^*_B$ respectively. There are two universal curves over $\tilde{K}^* = \tilde{K} \cap (\overline{M}^*_A \times \overline{M}^*_B)$ obtained via pull-back of the universal families U^*_A and U^*_B. These universal curves glue on the sections s_{pA} and s_{pB} to form a universal family

$$\tilde{\pi} : U^*_{\tilde{K}^*} \to \tilde{K}^*$$

of maps for the moduli problem of $\overline{M}$. It follows that

$$\omega_{U^*_{\tilde{K}^*}} \mid_{\tau^*_A(U^*_A)} = \tau^*_A(\omega_{\pi A}) + s_{pA},$$

$$\omega_{U^*_{\tilde{K}^*}} \mid_{\tau^*_B(U^*_B)} = \tau^*_B(\omega_{\pi B}) + s_{pB}.$$

Hence

$$\psi^*(\pi_*(c_1(\omega_\pi)^2)) = \tau^*_A(\pi_{A*}((c_1(\omega_{\pi A}) + s_{pA})^2)) + \tau^*_B(\pi_{B*}((c_1(\omega_{\pi B}) + s_{pB})^2)).$$

A normal bundle calculation yields $c_1(\omega_{\pi A}) \cdot s_{pA} = -s^2_{pA}$. Hence,

$$(c_1(\omega_{\pi A}) + s_{pA})^2 = c_1(\omega_{\pi A})^2 - s^2_{pA}$$

(similarly for B). Recall that $\pi_*(c_1(\omega_\pi)^2) = -\sum_{T \in \triangle} T$. Finally,

$$-\psi^*(K) = \sum_{T \in \triangle, T \neq K} \psi^*(T) \quad + \quad \tau^*_A(\pi_{A*}(c_1(\omega_{\pi A})^2 - s^2_{pA})) + \tau^*_B(\pi_{B*}(c_1(\omega_{\pi B})^2 - s^2_{pB})). \tag{13}$$

Lemmas (2.1.2), (2.2.1), and (2.2.2) express $\pi_{A*}(c_1(\omega_{\pi A})^2)$, $\pi_{A*}(s^2_{pA})$ explicitly in terms of the standard classes $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ on $\overline{M}_A$ (similarly for $\overline{M}_B$). Via equations (10) - (13), the ψ pull-back of every standard class $\{\mathcal{L}_i\} \cup \triangle \cup \{\mathcal{H}\}$ on $\overline{M}$ has now been expressed as the restriction to $\tilde{K}$ of a linear combination of the τ_A and τ_B pull-backs of standard classes on $\overline{M}_A$ and $\overline{M}_B$.

3.3. **The Algorithm.** The inductive algorithm for computing top intersection products is now clear. All top monomials in the elements $\{\mathcal{L}_i\} \cup \{\mathcal{H}\}$ are known by the First Reconstruction Theorem. If a monomial product on $\overline{M}$ includes a boundary class K, the intersection is carried out on $\tilde{K}$. By the above formulas (9)-(13), the desired monomial can be expressed as a sum of top products of standard classes on $\overline{M}_A$ and $\overline{M}_B$. Since $\overline{M}_A$ is of lesser degree or of lesser marking number than $\overline{M}$ (similarly for $\overline{M}_B$), the inductive process terminates.

3.4. **Characteristic Numbers.** Lemma (2.3.1) expresses the hyperplane tangency condition in terms of the standard classes. Hence all top products of the classes $\{\mathcal{L}_i\} \cup \{\mathcal{H}, \mathcal{T}\}$ can be effectively computed by the above algorithm.

It remains to check the top intersections of $\{\mathcal{L}_i\} \cup \{\mathcal{H}, \mathcal{T}\}$ are the characteristic numbers of rational curves. Let

$$c_1(\mathcal{L}_1)^{l_1} \cdots c_1(\mathcal{L}_n)^{l_n} \cdot \mathcal{H}^\alpha \cdot \mathcal{T}^\beta \tag{14}$$

be a top product on $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$. Since the $\mathcal{L}_i$ are pull-backs of $\mathcal{O}_{\mathbb{P}^r}(1)$ via the evaluation maps, codimension l_i linear spaces of $\mathbb{P}^r$ determine representatives of $c_1(\mathcal{L}_i)^{l_i}$. The cycle $\mathcal{H}^\alpha$ is determined by α codimension 2 linear spaces in $\mathbb{P}^r$. Finally, the cycle $\mathcal{T}^\beta$ is determined by β hyperplanes in $\mathbb{P}^r$. When $\beta \neq 0$, it is assumed that $d \geq 2$. The first step is to show that, for general choices of all the linear spaces of $\mathbb{P}^r$ in question, the intersection cycle (14) in $\overline{M}$ is at most 0 dimensional and corresponds (set theoretically) to the correct geometric locus. The second step is to show that the intersection cycle is multiplicity free.

Let $\mathbb{P}^{r*}$ be the parameter space of hyperplanes in $\mathbb{P}^r$. Define the universal tangency subvariety

$$\mathcal{T}_{univ} \subset \overline{M} \times \mathbb{P}^{r*}$$

as follows. Let $W_{univ} \subset \overline{M} \times \mathbb{P}^{r*}$ be the open locus of pairs $(\mu : C \to \mathbb{P}^r, H)$ where $\mu^{-1}(H)$ is a subscheme of d reduced points of $C_{nonsing}$. Let $\mathcal{T}_{univ}$ be the complement of W_{univ}. Let $\mathcal{T}_H$ be the the fiber of $\mathcal{T}_{univ}$ over the parameter point of the hyperplane H. $\mathcal{T}_H$ is exactly the tangency Weil divisor defined in section (2). Similarly, let

$$\mathcal{H}_{univ} \subset \overline{M} \times \mathbb{G}(\mathbb{P}^{r-2}, \mathbb{P}^r)$$

be the universal codimension 2 plane incidence subvariety. The fiber of $\mathcal{H}_{univ}$ over the parameter point of the codimension 2 plane P is $\mathcal{H}_P$. Let

$$I_{univ} \subset \overline{M} \ \times \ \mathbb{G}(r-l_1, r) \times \cdots \times \mathbb{G}(r-l_n, r) \ \times \ \mathbb{G}(r-2, r) \times \cdots \times \mathbb{G}(r-2, r)$$
$$\times \ \mathbb{P}^{r*} \times \cdots \times \mathbb{P}^{r*}$$

be the universal intersection cycle (14) defined by the universal divisors $\mathcal{T}_{univ}, \mathcal{H}_{univ}$ and the evaluation maps. I_{univ} is a closed subvariety.

In the first step, slightly more than the dimensionality of the general intersection cycle will be established. A map $\mu : C \to \mathbb{P}^r$ is *simply tangent* to a hyperplane H if

(i) $\mu^{-1}(H) \subset C_{nonsing}$, and
(ii) as a subscheme, $\mu^{-1}(H)$ consists of 1 double and $d-2$ reduced points.

A map $\mu : C \to \mathbb{P}^r$ has *simple intersection* with a codimension 2 plane P if

(i) $\mu^{-1}(P)$ consists of 1 point $x \in C_{nonsing}$, and
(ii) $Im(d\mu(x))$ and the tangent space of P span maximal rank.

Lemma 3.4.1. *For general choices of linear spaces*

$$L_1, \ldots, L_n,\ P_1, \ldots, P_\alpha,\ H_1, \ldots, H_\beta \tag{15}$$

the intersection cycle (14) *is at most* 0 *dimensional and set theoretically corresponds to maps* $\mu : C \to \mathbb{P}^r$ *satisfying:*

(1) $C \cong \mathbb{P}^1$, *μ is an immersion/embedding ($r = 2$ / $r \geq 3$).*
(2) $\forall k$, *μ is simply tangent to the hyperplanes* H_k.
(3) $\forall j$, *μ intersects the linear spaces* P_j *simply.*
(4) $\forall i$, *the μ-image of the* i^{th} *marked point lies in* L_i.

Proof. The intersection cycle I determined by the linear spaces (15) is the fiber of I_{univ} over the parameter points of the linear spaces. $dim(I) \leq 0$ is an open condition in the parameter space. It is first checked that general choice of the linear spaces (15) yields an intersection cycle of dimension at most 0.

Let $[\mu] \in \overline{M}$ be the moduli point of a map $\mu : C \to \mathbb{P}^r$. By Bertini's Theorem, the general hyperplane H is transverse to μ. Therefore, the general tangency divisor $\mathcal{T}_H$ satisfies $[\mu] \notin \mathcal{T}_H$. Similarly, the general incidence divisor $\mathcal{H}_P$ satisfies $[\mu] \notin \mathcal{H}_P$. By choosing at each stage tangency and incidence divisors that reduce the dimension of every component of the intersection, we see that

$$\mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta}$$

has codimension at least $\alpha + \beta$. Since the remaining intersections are obtained from basepoint free linear series, the general intersection cycle has dimension at most 0.

If the general parameter point yields an empty cycle I, there is nothing more to prove. Let W be the open set of the parameter space where $dim(I) = 0$. The conditions (1-3) on I determine open sets $W_1, W_2, W_3 \subset W$. Condition (4) is automatic. It suffices to show W_i is nonempty for $1 \leq i \leq 3$.

The subset $Y \subset \overline{M}$ of maps that are not immersion/embedding ($r = 2$ / $r \geq 3$) is of codimension at least 1. Hence, by the dimension reduction argument above, $Y \cap I = \emptyset$ for a general parameter point. Therefore, $W_1 \neq \emptyset$.

Let $W_{2,k}, W_{3,j} \subset W$ be the set of parameter points that satisfy condition (2), (3) for the hyperplane H_k linear space P_j respectively. Since $W_2 = \bigcap_{k=1}^{\beta} W_{2,k}$ and $W_3 = \bigcap_{j=1}^{\alpha} W_{3,j}$, it suffices to show that $W_{2,k}, W_{3,j} \neq \emptyset$. Let H_k be any hyperplane. The locus of moduli points $[\mu] \in \mathcal{T}_{H_k}$ that are not simply tangent is of codimension at least 2 in $\overline{M}$. By the dimension reduction argument, $W_{2,k} \neq \emptyset$. Similarly, the locus of moduli points $[\mu] \in \mathcal{H}_{P_j}$ that do not intersect simply is of codimension at least 2 in $\overline{M}$. As before, $W_{3,j} \neq \emptyset$. □

It must now be shown that the intersection cycle (14) is reduced for general linear spaces. This transversality is established by Kleiman's Bertini Theorem. Unfortunately, since the divisors $\mathcal{T}_H$, $\mathcal{H}_P$ need not move *linearly*, Bertini's Theorem cannot be directly applied to $\overline{M}$. Instead, an auxiliary construction is undertaken. Kleiman's Bertini Theorem is applied to the universal curve over $\overline{M}$. It will be shown that suitable transversality on the universal curve implies transversality on $\overline{M}$.

Let $\overline{M}^0 \subset \overline{M}$ be the open set of immersed/embedded ($r = 2$ / $r \geq 3$) maps with irreducible domains. Since (for general linear spaces) the intersection cycle (14) lies in $\overline{M}^0$, transversality need only be established in $\overline{M}^0$. Note that $\overline{M}^0$ is in the automorphism-free locus. Let $U \to \overline{M}^0$ be the universal curve. Let $\mu : U \to \mathbb{P}^r$ be the universal map. U and $\overline{M}^0$ are nonsingular. Let $\mathbb{P}T$ be the projective tangent bundle of $\mathbb{P}^r$. Since each point of $\overline{M}^0$ corresponds to an immersion/embedding, there is a natural algebraic map $\nu : U \to \mathbb{P}T$ given by the differential of μ. The map ν is a lifting of μ.

By projectivizing tangent spaces, the hyperplanes $H_1, \ldots, H_\beta$ define nonsingular, codimension 2 subvarieties of $\mathbb{P}T$:

$$\mathbb{P}H_1, \ldots, \mathbb{P}H_\beta$$

Let $U_1, \ldots, U_\beta$ be β copies of the universal curve U. Let $U'_1, \ldots, U'_\alpha$ be α more copies of U. Define the product

$$X \cong U'_1 \times_{\overline{M}^0} \ldots \times_{\overline{M}^0} U'_\alpha \times_{\overline{M}^0} U_1 \times_{\overline{M}^0} \ldots \times_{\overline{M}^0} U_\beta.$$

Let $\mu'_j : X \to \mathbb{P}^r$, $\nu_k : X \to \mathbb{P}T$ be the maps obtained by projection onto U'_j, U_k and composition with μ, ν respectively.

Kleiman's Bertini Theorem may now be applied. The group $GL_{r+1}(\mathbb{C})$ acts transitively on $\mathbb{P}^r$ and on $\mathbb{P}T$. Hence, the general intersection

$$\mu_1'^{\,-1}(P_1) \cap \ldots \cap \mu_\alpha'^{\,-1}(P_\alpha) \cap \nu_1^{-1}(\mathbb{P}H_1) \cap \ldots \cap \nu_\beta^{-1}(\mathbb{P}H_\beta) \subset X$$

is nonsingular and of the correct codimension (if nonempty).

It remains to obtain the corresponding result on $\overline{M}$. Consider the nonsingular, codimension 2 subvariety $\mu^{-1}(P_j) \subset U$. The projection $\mu^{-1}(P_j) \to \mathcal{H}_{P_j} \cap \overline{M}^0$ is étale and 1-1 over the locus of maps meeting P_j simply. Similarly, the projection $\nu^{-1}(\mathbb{P}H_k) \to \mathcal{T}_{H_k} \cap \overline{M}^0$ is étale and 1-1 over the the locus of maps simply tangent to H_k. From Lemma (3.4.2) below, the projection

$$\mu_1'^{\,-1}(P_1) \cap \ldots \cap \mu_\alpha'^{\,-1}(P_\alpha) \cap \nu_1^{-1}(\mathbb{P}H_1) \cap \ldots \cap \nu_\beta^{-1}(\mathbb{P}H_\beta)$$

$$\longrightarrow \mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0$$

is étale and 1-1 over the locus of points in $\mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0$ corresponding to simple intersection and tangency. It has therefore been proved that, for general linear spaces, the locus of $\mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0$ corresponding to simple intersection and tangency is nonsingular and of the correct codimension (if nonempty). It was shown above that, for general linear spaces, the intersection cycle (14) involves only maps that have simple intersection and tangency with the P_j and the H_k. Since the intersections $c_1(\mathcal{L}_i)^{l_i}$ are obtained from basepoint free linear series on $\overline{M}$, the further intersections yield a reduced intersection cycle (14) by Bertini's Theorem.

Lemma 3.4.2. *Let M be a nonsingular base. Let $\pi : U \to M$ be a smooth map of relative dimension 1. Let $D_1, D_2, \ldots, D_l \subset U$ be nonsingular, codimension 2 subvarieties such that D_i is étale and 1-1 over $\pi(D_i)$. Let $X \cong U_1 \times_M \ldots \times_M U_l$ be the fiber product of copies of U. Let $\rho_i : X \to U_i$ be the projection. Then*

$$\rho_1^{-1}(D_1) \cap \ldots \cap \rho_l^{-1}(D_l) \subset X$$

is étale and 1-1 over the intersection $\pi(D_1) \cap \ldots \cap \pi(D_l) \subset M$.

Proof. The issue is local on M. Let $m \in M$ be in the intersection of the $\pi(D_i)$. Choose local defining equations (f_i) of $\pi(D_i)$ near m. Let $u_i \in D_i$ be points over m. Locally (in the analytic topology) at u_i, U_i is an open set of the trivial product $\mathbb{C}_i \times M$ and D_i is the intersection of (f_i) with a section (z_i) of this product (z_i is the coordinate on $\mathbb{C}_i$). It now follows that local equations for $\rho_1^{-1}D_1 \cap \ldots \cap \rho_l^{-1}D_l$ at $(u_1, \ldots, u_l)$ are $(z_1, \ldots, z_l, f_1, \ldots, f_l)$ in $\mathbb{C}_1 \times \ldots \mathbb{C}_l \times M$, which is certainly étale over $(f_1, \ldots, f_l) \subset M$. □

All the characteristic numbers of rational curves in projective space can be algorithmically computed. For example, the number of twisted cubics in $\mathbb{P}^3$ through 2 points, 6 lines, and tangent to 2 planes can be expressed as

$$c_1(\mathcal{L}_1)^3 \cdot c_1(\mathcal{L}_2)^3 \cdot c_1(\mathcal{L}_3)^2 \cdots c_1(\mathcal{L}_8)^2 \cdot \mathcal{T}^2$$

on $\overline{M}_{0,8}(\mathbb{P}^3, 3)$ or

$$c_1(\mathcal{L}_1)^3 \cdot c_1(\mathcal{L}_2)^3 \cdot \mathcal{H}^6 \cdot \mathcal{T}^2$$

on $\overline{M}_{0,2}(\mathbb{P}^3, 3)$.

4. Examples

4.1. **Conics in $\mathbb{P}^2$ and $\mathbb{P}^3$.** Since the Hilbert schemes of lines and conics are Grassmanians and projective bundles over Grassmanians, the $\beta = 0$ characteristic numbers of rational curves in degrees 1 and 2 can be calculated directly via intersection theory on these Hilbert schemes. The tangency characteristic numbers for conics classically required the beautiful space of complete conics. $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is the space of complete conics. A new calculation of the characteristic numbers for plane conics is obtained by considering the pointed space $\overline{M}_{0,1}(\mathbb{P}^2, 2)$.

Let $\overline{M} = \overline{M}_{0,1}(\mathbb{P}^2, 2)$. $Pic(\overline{M}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$, $\mathcal{L}_1$, and the unique boundary component K corresponding to the partition $(\{1\} \cup \emptyset, 1 + 1 = 2)$. The top intersection numbers are $(dim\overline{M}_{0,1}(\mathbb{P}^2, 2) = 6)$:

$\mathcal{H}^6$	0	$\mathcal{H}^5 K$	0	$\mathcal{H}^4 K^2$	0
$\mathcal{H}^5\mathcal{L}_1$	+2	$\mathcal{H}^4 K\mathcal{L}_1$	+6	$\mathcal{H}^3 K^2\mathcal{L}_1$	+18
$\mathcal{H}^4\mathcal{L}_1^2$	+1	$\mathcal{H}^3 K\mathcal{L}_1^2$	+3	$\mathcal{H}^2 K^2\mathcal{L}_1^2$	+9

$\mathcal{H}^3 K^3$	0	$\mathcal{H}^2 K^4$	0	$\mathcal{H} K^5$	0	K^6	0
$\mathcal{H}^2 K^3\mathcal{L}_1$	−10	$\mathcal{H} K^4\mathcal{L}_1$	−30	$K^5\mathcal{L}_1$	+102		
$\mathcal{H} K^3\mathcal{L}_1^2$	−5	$K^4\mathcal{L}_1^2$	−15				

Note that $\mathcal{L}_1^3 = 0$. The line tangency class $\mathcal{T} = \frac{1}{2}(\mathcal{H} + K)$ is determined by Lemma (2.3.1). The characteristic number of plane conics through α points and tangent to β lines is $\frac{1}{2}\mathcal{H}^\alpha\mathcal{T}^\beta\mathcal{L}_1$:

$(1/2) \cdot \mathcal{H}^5\mathcal{L}_1$	1
$(1/2) \cdot \mathcal{H}^4\mathcal{T}\mathcal{L}_1$	2
$(1/2) \cdot \mathcal{H}^3\mathcal{T}^2\mathcal{L}_1$	4
$(1/2) \cdot \mathcal{H}^2\mathcal{T}^3\mathcal{L}_1$	4
$(1/2) \cdot \mathcal{H}\mathcal{T}^4\mathcal{L}_1$	2
$(1/2) \cdot \mathcal{T}^5\mathcal{L}_1$	1

The class of maps tangent to a fixed conic can easily be calculated by the methods of Lemma (2.3.1). Let $\mathcal{C} \in Pic(\overline{M}) \otimes \mathbb{Q}$ denote this conic tangency class. $\mathcal{C} = 3\mathcal{H} + K$. The number of plane conics tangent to 5 fixed conics is therefore $\frac{1}{2}\mathcal{C}^5\mathcal{L}_1 = 3264$.

For $r \geq 3$, $\overline{M}_{0,0}(\mathbb{P}^r, 2)$ differs from the space of complete conics and the algorithm described above yields a new computation of the characteristic numbers in these cases. Let $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^3, 2)$. $Pic(\overline{M}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$ and the unique boundary component K corresponding to the degree partition $1 + 1 = 2$. Also, $\tilde{K} \subset \overline{M}_{0,1}(\mathbb{P}^3, 1) \times \overline{M}_{0,1}(\mathbb{P}^3, 1)$. Since $\overline{M}_{0,1}(\mathbb{P}^3, 1)$ has no boundary, all top intersections are known. Using the formulas of section (3), the answers for the top intersections of $\mathcal{H}$ and K on $\overline{M}_{0,0}(\mathbb{P}^3, 2)$ $(dim\overline{M}_{0,0}(\mathbb{P}^3, 2) = 8)$ are:

$\mathcal{H}^8$	+92
$\mathcal{H}^7K$	+140
$\mathcal{H}^6K^2$	+140
$\mathcal{H}^5K^3$	−100
$\mathcal{H}^4K^4$	−68
$\mathcal{H}^3K^5$	+172
$\mathcal{H}^2K^6$	−20
$\mathcal{H}K^7$	−580
K^8	+1820

By Lemma (2.3.1), $\mathcal{T} = \frac{1}{2}(\mathcal{H} + K)$. The characteristic number of space conics through α lines and tangent to β planes is $\mathcal{H}^\alpha\mathcal{T}^\beta$:

$\mathcal{H}^8$	92
$\mathcal{H}^7\mathcal{T}$	116
$\mathcal{H}^6\mathcal{T}^2$	128
$\mathcal{H}^5\mathcal{T}^3$	104
$\mathcal{H}^4\mathcal{T}^4$	64
$\mathcal{H}^3\mathcal{T}^5$	32
$\mathcal{H}^2\mathcal{T}^6$	16
$\mathcal{H}\mathcal{T}^7$	8
$\mathcal{T}^8$	4

These characteristic numbers (with complete proofs) were known classically.

4.2. **Rational Plane Cubics.** Let $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^2, 3)$. $Pic(\overline{M}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$ and the unique boundary component K corresponds to the degree partition $1+2=3$. The algorithm described above yields the top intersections of $\mathcal{H}$ and K inductively. Since $\tilde{K} \subset \overline{M}_{0,1}(\mathbb{P}^2, 1) \times \overline{M}_{0,1}(\mathbb{P}^2, 2)$, first the top intersections on these Kontsevich spaces must be computed. $\overline{M}_{0,1}(\mathbb{P}^2, 1)$ has no boundary, hence all top products are known. There is a unique boundary component B of $\overline{M}_{0,1}(\mathbb{P}^2, 2)$, and $\overline{B} \subset \overline{M}_{0,2}(\mathbb{P}^2, 1) \times \overline{M}_{0,1}(\mathbb{P}^2, 1)$. Thus the top products on $\overline{M}_{0,2}(\mathbb{P}^2, 1)$ must be computed. Finally, the unique boundary component of $\overline{M}_{0,2}(\mathbb{P}^2, 1)$ requires knowledge of the top products on $\overline{M}_{0,3}(\mathbb{P}^2, 0)$ and $\overline{M}_{0,1}(\mathbb{P}^2, 1)$, which are known. The answers for the top intersections of $\mathcal{H}$ and K on $\overline{M}_{0,0}(\mathbb{P}^2, 3)$ $(dim\overline{M}_{0,0}(\mathbb{P}^2, 3) = 8)$ are:

$\mathcal{H}^8$	+12
$\mathcal{H}^7K$	+42
$\mathcal{H}^6K^2$	+129
$\mathcal{H}^5K^3$	+285
$\mathcal{H}^4K^4$	+336
$\mathcal{H}^3K^5$	−(2541/4)
$\mathcal{H}^2K^6$	−(8259/16)
$\mathcal{H}K^7$	+(19641/8)
K^8	−(44835/16)

Note that since K is $\mathbb{Q}$-Cartier, the intersections $\mathcal{H}^i \cdot K^j$ need not be integers. By Lemma (2.3.1), $\mathcal{T} = \frac{2}{3}(\mathcal{H} + K)$. The characteristic number of plane cubics through

α points and tangent to β lines is $\mathcal{H}^\alpha \mathcal{T}^\beta$:

$\mathcal{H}^8$	12
$\mathcal{H}^7 \cdot \mathcal{T}$	36
$\mathcal{H}^6 \cdot \mathcal{T}^2$	100
$\mathcal{H}^5 \cdot \mathcal{T}^3$	240
$\mathcal{H}^4 \cdot \mathcal{T}^4$	480
$\mathcal{H}^3 \cdot \mathcal{T}^5$	712
$\mathcal{H}^2 \cdot \mathcal{T}^6$	756
$\mathcal{H} \cdot \mathcal{T}^7$	600
$\mathcal{T}^8$	400

These characteristic numbers have been calculated by H. Zeuthen, S. Maillard, H. Schubert, G. Sacchiero, S. Kleiman, S. Speiser, and P. Aluffi. ([S], [Sa], [K-S], [A]). Complete proofs appear in [Sa], [K-S], and [A].

4.3. **Twisted Cubics in $\mathbb{P}^3$.** In case $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^3, 3)$, $Pic(\overline{M}) \otimes \mathbb{Q}$ is still generated freely by $\mathcal{H}$, K. A similar analysis yields the top intersections ($dim(\overline{M}) = 12$):

$\mathcal{H}^{12}$	$+80160$
$\mathcal{H}^{11}K$	$+121440$
$\mathcal{H}^{10}K^2$	$+148920$
$\mathcal{H}^9K^3$	$+112080$
$\mathcal{H}^8K^4$	-7824
$\mathcal{H}^7K^5$	-104100
$\mathcal{H}^6K^6$	$+35880$
$\mathcal{H}^5K^7$	$+(190095/2)$
$\mathcal{H}^4K^8$	$-(222855/2)$
$\mathcal{H}^3K^9$	$-(674007/16)$
$\mathcal{H}^2K^{10}$	$+(10112745/32)$
$\mathcal{H}K^{11}$	$-(5995065/8)$
K^{12}	$+(58086435/32)$

The hyperplane tangency class is again $\mathcal{T} = \frac{2}{3}(\mathcal{H} + K)$. The characteristic number of twisted cubics through α lines and tangent to β planes is $\mathcal{H}^\alpha \mathcal{T}^\beta$:

$\mathcal{H}^{12}$	80160
$\mathcal{H}^{11}\mathcal{T}$	134400
$\mathcal{H}^{10}\mathcal{T}^2$	209760
$\mathcal{H}^9\mathcal{T}^3$	297280
$\mathcal{H}^8\mathcal{T}^4$	375296
$\mathcal{H}^7\mathcal{T}^5$	415360
$\mathcal{H}^6\mathcal{T}^6$	401920
$\mathcal{H}^5\mathcal{T}^7$	343360
$\mathcal{H}^4\mathcal{T}^8$	264320
$\mathcal{H}^3\mathcal{T}^9$	188256
$\mathcal{H}^2\mathcal{T}^{10}$	128160
$\mathcal{H}\mathcal{T}^{11}$	85440
$\mathcal{T}^{12}$	56960

These characteristic numbers have been calculated by H. Schubert and others ([S], [K-S-X]). Complete proofs appear in [K-S-X].

4.4. **Rational Plane Quartics.** Let $\overline{M} = \overline{M}_{0,0}(\mathbb{P}^2, 4)$. Then $Pic(\overline{M}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$ and the boundary components J, K corresponding to the degree partitions 2+2 = 4, 1+3 = 4. The top intersection numbers are ($dim \overline{M}_{0,0}(\mathbb{P}^2, 4) = 11$):

$\mathcal{H}^{11}$	$+620$				
$\mathcal{H}^{10}K$	$+1620$	$\mathcal{H}^{10}J$	$+504$		
$\mathcal{H}^9K^2$	$+3564$	$\mathcal{H}^9JK$	$+1512$	$\mathcal{H}^9J^2$	$+0$
$\mathcal{H}^8K^3$	$+4052$	$\mathcal{H}^8JK^2$	$+4536$	$\mathcal{H}^8J^2K$	$+0$
$\mathcal{H}^7K^4$	-8340	$\mathcal{H}^7JK^3$	$+10920$	$\mathcal{H}^7J^2K^2$	$+672$
$\mathcal{H}^6K^5$	-48300	$\mathcal{H}^6JK^4$	$+15480$	$\mathcal{H}^6J^2K^3$	$+4320$
$\mathcal{H}^5K^6$	$+1260$	$\mathcal{H}^5JK^5$	-22296	$\mathcal{H}^5J^2K^4$	$+17184$
$\mathcal{H}^4K^7$	$+153300$	$\mathcal{H}^4JK^6$	-22728	$\mathcal{H}^4J^2K^5$	-11040
$\mathcal{H}^3K^8$	$-(338620/3)$	$\mathcal{H}^3JK^7$	$+70056$	$\mathcal{H}^3J^2K^6$	-34560
$\mathcal{H}^2K^9$	$-(13690660/27)$	$\mathcal{H}^2JK^8$	$+5880$	$\mathcal{H}^2J^2K^7$	$+51072$
$\mathcal{H}K^{10}$	$+(147582380/81)$	$\mathcal{H}JK^9$	-385560	$\mathcal{H}J^2K^8$	$+100800$
K^{11}	$-(278947820/81)$	JK^{10}	$+1310904$	J^2K^9	-616896
$\mathcal{H}^8J^3$	-364				
$\mathcal{H}^7J^3K$	-1260	$\mathcal{H}^7J^4$	$+630$		
$\mathcal{H}^6J^3K^2$	-3852	$\mathcal{H}^6J^4K$	$+1782$	$\mathcal{H}^6J^5$	-645
$\mathcal{H}^5J^3K^3$	-8836	$\mathcal{H}^5J^4K^2$	$+3588$	$\mathcal{H}^5J^5K$	$-(2385/2)$
$\mathcal{H}^4J^3K^4$	$+4980$	$\mathcal{H}^4J^4K^3$	-1788	$\mathcal{H}^4J^5K^2$	$+906$
$\mathcal{H}^3J^3K^5$	$+16356$	$\mathcal{H}^3J^4K^4$	-7830	$\mathcal{H}^3J^5K^3$	$+(8241/2)$
$\mathcal{H}^2J^3K^6$	-22060	$\mathcal{H}^2J^4K^5$	$+7770$	$\mathcal{H}^2J^5K^4$	-1815
$\mathcal{H}J^3K^7$	-46452	$\mathcal{H}J^4K^6$	$+22632$	$\mathcal{H}J^5K^5$	$-(22125/2)$
J^3K^8	$+255444$	J^4K^7	-92232	J^5K^6	$+28920$
$\mathcal{H}^5J^6$	$+(2419/8)$				
$\mathcal{H}^4J^6K$	$-(4743/8)$	$\mathcal{H}^4J^7$	$+(765/2)$		
$\mathcal{H}^3J^6K^2$	$-(18549/8)$	$\mathcal{H}^3J^7K$	$+1305$	$\mathcal{H}^3J^8$	$-(5649/8)$
$\mathcal{H}^2J^6K^3$	$-(3455/8)$	$\mathcal{H}^2J^7K^2$	$+(1923/2)$	$\mathcal{H}^2J^8K$	$-(6615/8)$
$\mathcal{H}J^6K^4$	$+(39075/8)$	$\mathcal{H}J^7K^3$	-1680	$\mathcal{H}J^8K^2$	$+(2163/8)$
J^6K^5	$-(56631/8)$	J^7K^4	$+(1701/2)$	J^8K^3	$+(2289/8)$
$\mathcal{H}^2J^9$	$+(4375/8)$				
$\mathcal{H}J^9K$	$+189$	$\mathcal{H}J^{10}$	$-(7875/32)$		
J^9K^2	-189	$J^{10}K$	$+0$	J^{11}	$+(10143/128)$

The line tangency class is $\mathcal{T} = \frac{3}{4}\mathcal{H} + J + \frac{3}{4}K$. The characteristic number of rational plane quartics through α points and tangent to β lines is $\mathcal{H}^\alpha \mathcal{T}^\beta$:

$\mathcal{H}^{11}$	620
$\mathcal{H}^{10}\mathcal{T}$	2184
$\mathcal{H}^9\mathcal{T}^2$	7200
$\mathcal{H}^8\mathcal{T}^3$	21776
$\mathcal{H}^7\mathcal{T}^4$	59424
$\mathcal{H}^6\mathcal{T}^5$	143040
$\mathcal{H}^5\mathcal{T}^6$	295544
$\mathcal{H}^4\mathcal{T}^7$	505320
$\mathcal{H}^3\mathcal{T}^8$	699216
$\mathcal{H}^2\mathcal{T}^9$	783584
$\mathcal{H}^1\mathcal{T}^{10}$	728160
$\mathcal{T}^{11}$	581904

The characteristic numbers of rational plane quartics were calculated long ago by H. Zeuthen in [Z].

4.5. **Cuspidal Rational Plane Curves.** For $d \geq 1$, let N_d be the number of irreducible, nodal rational plane curves passing through $3d-1$ general points in $\mathbb{P}^2$. N_d is a $\beta = 0$ characteristic number. The numbers N_d satisfy a beautiful recursion relation ([K-M]):

$$N_1 = 1,$$

$$\forall d \geq 2, \quad N_d = \sum_{i+j=d,\ i,j>0} N_i N_j i^2 j \left(j \binom{3d-4}{3i-2} - i \binom{3d-4}{3i-1} \right).$$

The first few N_d's are:

$$N_1 = 1,\ N_2 = 1,\ N_3 = 12,\ N_4 = 620,\ N_5 = 87304,\ N_6 = 26312976,\ \ldots .$$

As a final application, the enumerative geometry of cuspidal rational plane curves is considered. A rational plane curve C is *1-cuspidal* if the singularities of C consist of nodes and exactly 1 cusp. For $d \geq 3$, let C_d be the number of irreducible, 1-cuspidal rational plane curves passing through $3d-2$ general points in $\mathbb{P}^2$.

Proposition 1. *The numbers C_d can be expressed in terms of the N_d:*

$$\forall d \geq 3, \quad C_d = \frac{3d-3}{d} N_d \ + \ \frac{1}{2d} \cdot \sum_{i=1}^{d-1} \binom{3d-2}{3i-1} N_i N_{d-i} \big(3i^2(d-i)^2 - 2di(d-i) \big).$$

The first few C_d's are:

$$C_3 = 24,\ C_4 = 2304,\ C_5 = 435168,\ C_6 = 156153600,\ \ldots .$$

C_3 is the degree of the locus of cuspidal cubics. C_4 has been computed by H. Zeuthen ([Z]). The 1-cuspidal numbers C_d are evaluated by intersecting divisors on $\overline{M}_{0,0}(\mathbb{P}^2, d)$.

Let $d \geq 3$. Let $M_{0,0}(\mathbb{P}^2, d)$ be $\overline{M}_{0,0}(\mathbb{P}^2, d)$ minus the boundary. Let $Z \subset M_{0,0}(\mathbb{P}^2, d)$ be the subvariety of maps that are not immersions. It is easily seen that Z is of pure codimension 1 and the generic element of every component corresponds to a 1-cuspidal rational plane curve. Let $\mathcal{Z}$ be the Weil divisor obtained by the closure of Z in $\overline{M}_{0,0}(\mathbb{P}^2, d)$. By the dimension reduction argument of section (3.4), the intersection cycle on $\overline{M}_{0,0}(\mathbb{P}^2, d)$

$$\mathcal{Z} \cap \mathcal{H}^{3d-2} \tag{16}$$

determined by general points $P_1, \ldots, P_{3d-2}$ is of dimension (at most) 0 and lies in Z. A simple modification of the corresponding argument in section (3.4) can be applied to show that (16) is reduced for general choices of P_j. Hence $C_d = \mathcal{Z} \cdot \mathcal{H}^{3d-2}$.

The boundary of $\overline{M}_{0,0}(\mathbb{P}^2, d)$ simply consists of the $[\frac{d}{2}]$ Weil divisors K^i ($1 \leq i \leq [\frac{d}{2}]$). Recall that K^i is the boundary component corresponding to the degree partition $i + (d-i) = d$. By Lemmas (1.2.1-1.2.2), the elements $\{\mathcal{H}\} \cup \{K^i\}$ span a basis of $Pic(\overline{M}_{0,0}(\mathbb{P}^2, d)) \otimes \mathbb{Q}$.

Lemma 4.5.1. *The class of $\mathcal{Z}$ in $Pic(\overline{M}_{0,0}(\mathbb{P}^2, d)) \otimes \mathbb{Q}$ is determined by ($d \geq 3$)*

$$\mathcal{Z} = \frac{3d-3}{d} \mathcal{H} \ + \ \sum_{i=1}^{[\frac{d}{2}]} \frac{3i(d-i) - 2d}{d} K^i \ . \tag{17}$$

Proof. Let S, μ, $\overline{S}$, $\lambda : C \to \overline{M}_{0,0}(\mathbb{P}^2, d)$ be exactly as in the proof of Lemma (2.3.1). It will be checked that the left and right sides of (17) have the same intersection with C. As before, a point of the intersection $C \cdot K^i$ can arise in two cases. A simple point of degree i or $d-i$ can be blown-up. Let

$$C \cdot K^i = x_i + y_i,$$

where x_i and y_i are the number of instances of the first and second case respectively. Let E_{x_i} be the union of the x_i exceptional divisors in $\overline{S}$ obtained from the x_i points of $C \cdot K^i$. Let E_{y_i} be defined similarly.

First, the intersection $C \cdot \mathcal{Z}$ is calculated. Consider $\overline{\mu} : \overline{S} \to \mathbb{P}^2$; then $\overline{\mu}^*(\mathcal{O}_{\mathbb{P}}(1))$ is the element $(d,k) - \sum iE_{x_i} - \sum (d-i)E_{y_i}$. The differential map yields an injection of *sheaves*:

$$0 \rightarrow T_{\overline{S}} \overset{d\overline{\mu}}{\rightarrow} \overline{\mu}^*(T_{\mathbb{P}^2}) \rightarrow Q \rightarrow 0. \tag{18}$$

For general maps $\overline{\mu}$, Q is a line bundle supported on a nonsingular curve D. The restriction of the sequence (18) to D yields an exact sequence of *bundles* on D:

$$0 \rightarrow L \rightarrow T_{\overline{S}}|_D \overset{d\overline{\mu}}{\rightarrow} \overline{\mu}^*(T_{\mathbb{P}^2})|_D \rightarrow Q|_D \rightarrow 0, \tag{19}$$

where L is a line bundle on D. Finally, there is an exact sequence of bundles on D obtained from the projection $\overline{\pi} : \overline{S} \to C$:

$$0 \rightarrow V \rightarrow T_{\overline{S}}|_D \overset{d\overline{\pi}}{\rightarrow} \overline{\pi}^*(T_C) \rightarrow 0, \tag{20}$$

where V is a line bundle on D. Maps in the family $\overline{\pi}$ have zero differential exactly at the points of intersection $\mathbb{P}(V) \cdot \mathbb{P}(L) \subset \mathbb{P}(T_{\overline{S}}|D)$. Hence

$$C \cdot \mathcal{Z} = \mathbb{P}(V) \cdot \mathbb{P}(L)\ .$$

A lengthy, routine exercise in Chern classes and exact sequences now yields

$$C \cdot \mathcal{Z} = (6d-6)k + \sum_{i=1}^{[\frac{d}{2}]} (-3i^2 + 3i - 2)x_i + \sum_{i=1}^{[\frac{d}{2}]} (-3(d-i)^2 + 3(d-i) - 2)y_i.$$

Algebraic manipulation and the relation

$$C \cdot \mathcal{H} = 2dk - \sum_{i=1}^{[\frac{d}{2}]} i^2 x_i - \sum_{i=1}^{[\frac{d}{2}]} (d-i)^2 y_i$$

yield the result. □

It remains to compute $\mathcal{Z} \cdot \mathcal{H}^{3d-2}$. By Lemma (4.5.1), it suffices to determine the products $K^i \cdot \mathcal{H}^{3d-2}$. If $i \neq d/2$, the result

$$K^i \cdot \mathcal{H}^{3d-2} = \binom{3d-2}{3i-1} i(d-i) N_i N_{d-i}$$

is obtained from a simple geometric argument. In case $i = d/2$, division by 2 is required to account for symmetry:

$$K^{\frac{d}{2}} \cdot \mathcal{H}^{3d-2} = \frac{1}{2} \binom{3d-2}{3\frac{d}{2}-1} (\frac{d}{2})^2 N_{\frac{d}{2}}^2\ .$$

Evaluation of $\mathcal{Z} \cdot \mathcal{H}^{3d-2}$ yields the formula for C_d. The proof of Proposition (1) is complete.

References

[Al] V. Alexeev, *Moduli Spaces $M_{g,n}(W)$ for Surfaces*, in *Higher-dimensional Complex Varieties* (*Trento, 1994*), de Gruyter, Berlin, 1996, pp. 1–22. CMP 97:16

[A] P. Aluffi, *The Enumerative Geometry of Plane Cubics II: Nodal and Cuspidal Cubics*, Math. Ann., 289 (1991), 543-572. MR **92f:**14055

[D-I] P. Di Francesco and C. Itzykson, *Quantum Intersection Rings*, in *The Moduli Space of Curves*, R. Dijkgraaf, C. Faber, and G. van der Geer, eds., Birkhäuser, 1995, pp. 81-148. MR **96k:**11041a

[FP] W. Fulton and R. Pandharipande, *Notes on Quantum Cohomology and Stable Maps*, in *Algebraic Geometry—Santa Cruz, 1995*, Proc. Sympos. Pure Math., vol. 62, part 2, Amer. Math. Soc., Provicence, RI, 1997, pp. 45–96. MR **98h:**14003

[Ke] S. Keel, *Intersection Theory of Moduli Spaces of Stable n-Pointed Curves of Genus* 0, Trans. AMS, 330 (1992), 545-574. MR **92f:**14003

[K] M. Kontsevich, *Enumeration of Rational Curves Via Torus Actions*, in *The Moduli Space of Curves*, R. Dijkgraaf, C. Faber, and G. van der Geer, eds., Birkhäuser, 1995, pp. 335-368. MR **97d:**14077

[K-M] M. Kontsevich and Y. Manin, *Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry*, Commun. Math. Phys. 164 (1994) 525-562. MR **95i:**14049

[K-S] S. Kleiman, R. Speiser, *Enumerative Geometry of Nodal Plane Curves*, Algebraic Geometry - Sundance 1986, Lecture Notes in Mathematics 1311, Springer-Verlag: Berlin, 1988, pp. 156–196. MR **89k:**14095

[K-S-X] S. Kleiman, S. Stromme, S. Xambo, *Sketch of a Verification of Schubert's Number 5,819,539,753,680 of Twisted Cubics*, in *Space Curves* (Roca di Papa, 1985), Lecture Notes in Mathematics 1266, Springer-Verlag: Berlin, 1987, pp. 156–180. MR **88k:**14035

[P] R. Pandharipande, *The Canonical Class of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and Enumerative Geometry*, Internat. Math. Res. Notices **1997**, no. 4, 173–186. MR **98h:**14067

[R-T] Y. Ruan and G. Tian, *A Mathematical Theory of Quantum Cohomology*, J. Diff. Geom 42 (1995), 259-367. MR **96m:**58033

[Sa] G. Saccheiro, *Numeri Caratteristici delle Cubishe Piane Nodali*, preprint 1985.

[S] H. Schubert, *Kalkül der Abzählenden Geometrie*, B. G. Teubner: Leipzig, 1879.

[Z] H. Zeuthen, *Almindelige Egenskaber*, Danske Videnkabernes Selskabs Skrifter-Natur og Math, 10 (1873).

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Current address: Department of Mathematics, California Institute of Technology, Pasadena, California 91125

E-mail address: `rahulp@cco.caltech.edu`